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The Geometry of Generalized Quantum Logics

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Let Π be a quantum logic; by this we mean an orthocomplemented, orthomodular, partially ordered set. We assume that Π carries a sufficiently large collection Δ of states (probability measures). Then, Δ is embedded as a base for the cone of a partially ordered normed space $\mathscr{S}$ and Π is also embedded in the dual order-unit Banach space $\mathscr{S}^*$. We consider conditions on the pairs (Δ, Π) and $(\mathscr{S}, \mathscr{S}^*)$ that guarantee that Π is a dense subset of the extreme points of the positive part of the unit ball of $\mathscr{S}^*$. We demonstrate a connection of these conditions in noncommutative measure theory. The assumptions made here are far weaker than the assumptions of the traditional quantum mechanical formalisms and also apply to situations quite different from quantum mechanics. Finally, we show the connections of this theory to the well-known models of quantum mechanics and classical measure theory.

1. INTRODUCTION

The theory of empirical logics, or generalized quantum logics, has recently been introduced and developed by Foulis and Randall (1972, 1973a, b). In these papers, they introduced the notion of "a manual of operations," which is a formal mathematical structure describing very general laboratory procedures. Using these manuals they define the concepts of state, observable, proposition, logic, etc. These physical situations are thus very general and contain as special examples classical Boolean logic and Hilbert space quantum mechanics. More importantly, this theory contains many examples that are very different from quantum mechanics; yet, in these examples simultaneity of measurement and uncertainty can be described with reasonable proficiency. Since the results in this paper concern the geometric and linear topological properties of quantum logics embedded in certain Banach spaces, we will not begin with the foundation concept of a manual. Rather, we will assume at the outset that our logic of propositions is given to us and that a sufficiently large collection of probability measures (states) on this logic is also provided. The reader is urged to look at the papers of Randall and Foulis already mentioned to see how the notions of logic and state are obtained from the more fundamental notion of a manual.

Let us now state the problem we are interested in describing in this paper. Let Π denote our logic and Δ our selected convex set of probability measures on Π. All the definitions and algebraic assumptions for Π and Δ that are necessary for the following discussion will be given with precision in the next section. Let $\mathscr{S}$ be the linear space of real-valued functions on Π spanned by Δ. Then Δ is a base for the positive cone of $\mathscr{S}$ and $\mathscr{S}$ is a base-normed space. The logic Π can be embedded in $\mathscr{S}^*$, the partially ordered, order-unit Banach space dual to $\mathscr{S}$. The problem is this: Under what simple mathematical conditions can we identify Π with the extreme points of the positive part of the unit ball of $\mathscr{S}^*$? Let us now discuss the problem in various situations for which the solution is known.

Let $\mathscr{H}$ be a separable Hilbert space; Π is the orthomodular lattice of projections on $\mathscr{H}$ and Δ is taken to be the convex set of positive trace class operators with unit trace, see e.g., Jauch (1968), Chapter 8. $\mathscr{S}$ is the real linear space of self-adjoint trace class operators and $\mathscr{S}^*$ is identified with the order-unit Banach space of bounded self-adjoint operators on $\mathscr{H}$. Kadison (1951, p. 328) proved that the set of extreme points of the positive part of the unit ball of $\mathscr{S}^*$ is the set of projections on $\mathscr{H}$. In fact, he proved much more, namely, that the set of idempotents in a von Neumann algebra forms the extreme boundary of the positive part of the unit ball in the algebra. This example will be explained more fully in the last section in this article. Recently Alfsen and Shultz (1974), constructing a geometric spectral theory, proved that the projective units of a certain order-unit Banach space correspond to the extreme points of the positive part of its unit ball. This includes Kadison's result when these projective units are identified with the algebra's idempotents; further, this theory also applies to general Jordan–Banach algebras. These latter algebras need not arise from algebras of operators on a Hilbert space. Each of these examples has one common feature, namely, each element of the positive cone (of the W^* algebra or the order-unit space) is represented by a spectral integral where the range of the measure defined in the integral is contained in the extreme points of the positive part of the unit ball. As we know in the standard quantum mechanical formalism, the bounded observables (positive linear functionals on the normal states) are identified with the spectral measures. If we now assume the quantum logic is *finite* and some other mild hypotheses are satisfied, Rüttimann (1977) has been able to identify the logic with the extreme points of the positive part of the unit ball in S^*. In this work, there is no assumption of spectral representation in $\mathscr{S}^*$. Along this line, the physically motivated examples given by Foulis and Randall which also have finite logics have positive linear functionals on their spaces of states which are not determined by any spectral measure. The interpretation here is simple: there are positive linear functionals on these very general state spaces which are not associated with any bounded observable of the physical situation being described. These functionals seem to arise rather naturally whenever the geometry of the set of probability states contained in $\mathscr{S}$ is very different from that of a von Neumann algebra or Banach lattice. Hence, these physical situations, displaying a statistical uncertainty, are very different from classical statistics and also from traditional Hilbert space quantum mechanics. Therefore, the real problem to be tackled here is to find a simple mathematical property that applies to infinite logics, that is less restrictive than the assumption that each positive linear functional on the space of states be represented by a proposition-valued measure (a spectral integral), and that produces the result that the logic can be identified with the extreme points of the positive part of the unit ball in $\mathscr{S}^*$. This has been nearly accomplished in Theorem 10.

In the final part of this Introduction let us connect these results with some of the existing literature and also give some additional justification for proving these results at this level of generality. The finite logics given by Randall and Foulis and Rüttimann, already mentioned, are generally not projective geometries. Thus, they appear to be very different for the original examples proposed by Birkhoff and von Neumann (1936). In 1967, Gunson gave an algebraic formulation of quantum mechanics in which a picture close to the Hilbert space formalism was obtained. The fundamental work of Ludwig (1970) established many of the mathematical formalities used in this paper. In particular, the propositions of our logic are the decision effects in Ludwig's system. He proved that these points were also extreme points as we do in Theorem 10. In both of these works the aim is to derive the Hilbert space formalism from physically motivated axioms. This is not our purpose here, since we maintain that there exist other systems displaying some of the uncertainty properties of quantum mechanics while not admitting Hilbert space representations. In a forthcoming paper, Rüttimann and Cook (to appear) will present a theory of symmetry groups on quantum logics in which each symmetry on the states (a convex automorphism of Δ) is induced by and induces a symmetry on the logic (an automorphism of Π), see Jauch (1968, p. 142). For this result, it is necessary to give a geometric characterization of the elements of Π as extreme points of the positive part of the unit ball in $\mathscr{S}^*$. With this result, it is then clear that the group of symmetries on the states is isomorphic to the group of automorphisms of the logic.

2. PRELIMINARIES

Let $(\Pi, \leqslant)$ be a partially ordered set with at least two elements, largest element e and smallest element 0. Let Π be equipped with an orthocomplementation $'$. Specifically, if $p \in \Pi$ then $p' \in \Pi$ and the following are satisfied:

(i) $(p')' = p$ for all $p \in \Pi$;
(ii) $p \leqslant q$ in Π implies $q' \leqslant p'$;
(iii) $p \vee p' = e$ and $p \wedge p' = 0$ for all $p \in \Pi$.

We say that p, q are *orthogonal*, denoted $p \perp q$, if $p \leqslant q'$. We will assume that if $\{p_1, p_2, \ldots, p_n\}$ is a finite orthogonal set in Π then $\vee_{k=1}^{n} p_k$ exists in Π. Finally, we insist that $(\Pi, \leqslant)$ satisfies the *orthomodular identity*: $p \leqslant q$ in Π implies $q = p \vee (q \wedge p')$. The set Π satisfying all the conditions above will be called a *quantum logic* and its elements *propositions*. $(\Pi, \leqslant)$ is called σ *orthocomplete* if for each countable orthogonal sequence $\{p_1, p_2, \ldots\}$ in Π there exists $p \in \Pi$ such that $p = \bigvee_{k=1}^{\infty} p_k$. We note here that a quantum logic need not be a lattice. For further details on proposition systems, the reader is urged to see Jauch (1968), Chapter 5, or Piron (1976), Chapter 2.

A *state* $\omega: \Pi \to [0, 1]$ is a function with the properties: (i) $\omega(0) = 0$ and $\omega(e) = 1$; (ii) if $\{p_1, p_2, \ldots, p_n\}$ is a finite orthogonal set in Π with $p = \bigvee_{k=1}^{n} p_k$, then $\omega(p) = \sum_{k=1}^{n} \omega(p_k)$. A state ω is called *countably additive* if $\{p_1, p_2, \ldots\}$ is an orthogonal sequence with $p = \bigvee_{k=1}^{\infty} p_k$ in Π then $\omega(p) = \sum_{k=1}^{\infty} \omega(p_k)$. We will denote with Ω the set of all states (finitely additive) on Π. We will also assume that there are sufficiently many states in Ω to separate the points of Π; i.e., for $p \neq q$ in Π, there exists $\omega \in \Omega$ such that $\omega(p) \neq \omega(q)$. Finally, it is easy to see that Ω forms a convex set.

As in measure theory, one can form linear combinations of positive measures and thus construct vector spaces of signed measures. We now do likewise; let $\mathscr{C} = \{\alpha\omega: \omega \in \Omega, \alpha \geqslant 0 \text{ in } \mathbb{R}\}$ and $\mathscr{S} = \mathscr{C} - \mathscr{C}$. Then $\mathscr{S}$ is a vector space, called the space of signed states, with $\mathscr{C}$ as a generating cone. It is easy to see that Ω forms a base for $\mathscr{C}$; specifically, for each $x \neq 0$ in $\mathscr{C}$ there is a unique scalar $\alpha > 0$ and a unique ω in Ω with $x = \alpha\omega$. If Δ is a convex subset of Ω, which separates the points of Π, then we can form the subcone $\mathscr{C}_\Delta$ of $\mathscr{C}$ with base Δ and construct the analogous vector space $\mathscr{S} = \mathscr{C}_\Delta - \mathscr{C}_\Delta$. If U is the convex hull of $\Delta U(-\Delta)$ then U is a symmetric, absorbing, and convex set in $\mathscr{S}$. Its Minkowski functional is a seminorm on $\mathscr{S}$, see, e.g., Kelley and Namioka (1963, p. 15). This functional is, in fact, a norm—called the *base norm*. We denote it by $\|\cdot\|$ and observe for each $s \in \mathscr{S}$ that $\|s\| = \inf\{\alpha + \beta: \alpha\omega - \beta\nu, \alpha, \beta \geqslant 0 \text{ in } \mathbb{R}, \omega, \nu \in \Delta\}$.

Let $\mathscr{S}^*$ be the continuous dual of $(\mathscr{S}, \|\cdot\|)$. Each p in Π is represented by an element $\hat{p}$ of $\mathscr{S}^*$ by the formula $\hat{p}(\omega) = \omega(p)$ for each $\omega \in \Delta$. It is easy to see that this formula defines a unique element in $\mathscr{S}^*$ and, therefore, one may consider Π embedded in $\mathscr{S}^*$. It is well known that $\mathscr{S}^*$ is an order-unit Banach space (Alfsen, 1971, p. 18) and the following question now arises: Is the embedding of Π in $\mathscr{S}^*$ order preserving and injective? We call Δ *full* over Π when $p_1 \leqslant p_2$ in Π iff $\omega(p_1) \leqslant \omega(p_2)$ for all $\omega \in \Delta$. Since we want the embedding of Π in $\mathscr{S}^*$ to be injective and order preserving, we will always assume Δ is full over Π. This clearly yields these conditions. Further, when Δ is full over Π, Δ separates the elements of Π. It now follows that $e \in \Pi$ is represented by the order-unit in $\mathscr{S}^*$ and the order interval $[-e, e]$ in $\mathscr{S}^*$ is the closed unit ball of $\mathscr{S}^*$. Lastly, $p \in \Pi$ implies $p' = e - p$ in $\mathscr{S}^*$.

3. THE JORDAN DECOMPOSITION PROPERTY

Let $\mathscr{P} \subseteq \Pi$ such that $0, e \in \mathscr{P}$. For $s \in \mathscr{S}$ we define

$$\|s\|_Z = \sup \{s(p) - s(p') : p \in \mathscr{P}\}$$

Since $p - p' = 2p - e$ in $\mathscr{S}^*$, we have $\|s\|_Z = \sup \{2s(p) - s(e) : p \in \mathscr{P}\}$. We use a subscript Z on this function because the Jordan decomposition property for states was introduced by Zierler (1959, p. 21) and this is intimately connected with $\|\cdot\|_Z$. It is easy to see that $\|\cdot\|_Z$ is a seminorm on $\mathscr{S}$ and agrees with the base norm on $\mathscr{C}$. Since $-e \leqslant p - p' \leqslant e$, for each s in $\mathscr{P}$, $\|s\|_Z \leqslant \|s\|$.

A set of states $\Delta \subseteq \Omega$ is called *unital* if for each $p \in \Pi$ there is an ω in Δ such that $\omega(p) = 1$. Further, when Δ is unital over Π, for each $p \in \Pi$, $p - p'$ has order-unit norm equal to unity in $\mathscr{S}^*$.

If E is a normed space, a family of functionals $F = \{f : f \in E^*\}$ is called *total* over E if $f(x) = 0$ for all $f \in F$ implies $x = 0$ in E. It is well known that F is total over E iff span (F) is $w(E^*, E)$-dense in E^*. The following simple proposition is essential.

> *1. Proposition.* The seminorm $\|\cdot\|_Z$ is a norm iff $\mathscr{P} \subseteq \Pi$ is total over $\mathscr{S}$.

Proof. Assume $\mathscr{P}$ is total and for some $s \in \mathscr{S}$ $\|s\|_Z = 0$. Then, for all $p \in \mathscr{P}$, $s(p - p') = 0$; this implies $2s(p) - s(e) = 0$ or that $s(p) = \frac{1}{2}s(e)$. Since this is true for all $p \in \mathscr{P}$ and $0 \in \mathscr{P}$, we have $s(e) = 0$. Therefore, $s(p) = 0$ for all $p \in \mathscr{P}$ and from totality s must be 0.

The converse is clear. ∎

We observe that Π is always total over $\mathscr{S}$. Let $\mathscr{P} \subseteq \Pi$ and let Δ be a set of states on Π. We call $\mathscr{P}$ Δ *fundamental* provided: (i) $0, e \in \mathscr{P}$, (ii) $\{p_1, p_2, \ldots, p_n\} \subseteq \mathscr{P}$ implies $p_1 \vee p_2 \vee \cdots \vee p_n \in \mathscr{P}$, and (iii) for each $q \in \Pi$, $\omega \in \Delta$ and $\epsilon > 0$, there exists $p \in \mathscr{P}$ such that $p \leqslant q$ and $\omega(q - p) < \epsilon$.

> *2. Proposition.* If Δ is a convex set that separates the points of Π, and $\mathscr{P}$ is Δ fundamental in Π, then $\mathscr{P}$ is total over $\mathscr{S}$ and, therefore, $\|\cdot\|_Z$ is a norm on $\mathscr{S}$.

Proof. Suppose $s \in \mathscr{S}$ and $s(p) = 0$ for all $p \in \mathscr{P}$. Since Δ is a base for the positive cone of $\mathscr{S}$, $s = \alpha\omega - \beta\nu$. Then, $\alpha\omega(p) = \beta\nu(p)$ for all $p \in \mathscr{P}$ and, in particular, $\alpha\omega(e) = \beta\nu(e)$. Assuming without loss that $\alpha \neq 0$, we have $\alpha = \beta \neq 0$. Again assuming without loss that $\omega \neq \nu$, there exists $q \in \Pi$, $q \neq 0$ and $\omega(q) \neq \nu(q)$. Let $\epsilon = |\omega(q) - \nu(q)| > 0$. Since $\mathscr{P}$ is Δ fundamental, there exists $p \in \mathscr{P}$ such that $p \leqslant q$, $|\omega(q) - \omega(p)| < \epsilon/2$ and $|\nu(p) - \nu(q)| < \epsilon/2$. Hence, $|\omega(q) - \nu(q)| \leqslant |\omega(q) - \omega(p)| + |\omega(p) - \nu(p)| + |\nu(p) - \nu(q)| < \epsilon$. Having a contradiction, we must have $s = 0$. ∎

The connection of this concept with regular measures is explained in the last section. The following proposition is essentially contained in Kronfli (1970, p. 195) and revised by Gudder (1973, p. 206). Our proof is entirely different and rather simple. We will not use the result in the sequel, but it seems to this author to be interesting in itself.

> *3. Proposition.* Let Π be a σ-complete lattice, Δ the set of all countably additive states on Π, and let Δ separate the points of Π. If $\mathscr{P} \subseteq \Pi$ is Δ fundamental then Δ with the topology induced from the norm $\|\cdot\|_Z$ is a complete metric space.

Proof. Let (ω_n) be a $\|\cdot\|_Z$-Cauchy sequence in Δ. Let $p \in \Pi$ and $\epsilon > 0$ be given. There exists an index N such that $n, m \geqslant N$ implies $\|\omega_n - \omega_m\|_Z < \epsilon/3$. Since $\mathscr{P}$ is fundamental, there exists q in $\mathscr{P}$ (depending on n and m) such that $q \leqslant p$ and

$$|\omega_n(p) - \omega_m(p)| \leqslant \omega_n(p - q) + \|\omega_n - \omega_m\|_Z\|q\| + \omega_m(p - q)$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

Therefore, define

$$\omega(p) = \lim_{n\to\infty} \omega_n(p)$$

for each $p \in \Pi$. It is easy to see that ω has range in [0, 1] and is finitely additive. To obtain countable additivity consider the following: Let (p_k) be an orthogonal sequence in Π with $p = \bigvee_k p_k$. Let $\mathscr{A}$ be the σ sublattice of Π generated by the set $\{p, p_1, p_2, \ldots\}$. If we restrict each ω_n to $\mathscr{A}$ then ω_n is a countably additive measure on $\mathscr{A}$ and $\omega_n(S) \leqslant 1$ for all n and all S in $\mathscr{A}$. It is well known that the event-wise limit of uniformly bounded measures is a finite countably additive measure, see, e.g., Royden (1968, p. 232). Thus, ω is a measure on $\mathscr{A}$ and, consequently, $\omega \in \Delta$. Finally, $(\omega_n) \to \omega$ in $\|\cdot\|_Z$. Since Π is total and (ω_n) is norm-Cauchy, this follows immediately. ∎

If $s_1, s_2 \geqslant 0$ in $\mathscr{S}$ then we say that s_1 and s_2 are *$\mathscr{P}$-mutually singular* if there exists p in $\mathscr{P}$ such that $s_1(p) = 0$ and $s_2(p') = 0$. $\mathscr{S}$ is said to be *$\mathscr{P}$-Jordan generated* if for each s in $\mathscr{S}$ there exist vectors $u, v \geqslant 0$ in $\mathscr{S}$ for which $s = u - v$ and u, v are $\mathscr{P}$-mutually singular. Finally, we say that $\mathscr{S}$ is *ϵ–$\mathscr{P}$-Jordan generated* if for each s in $\mathscr{S}$ and $\epsilon > 0$ there exist vectors $u, v \geqslant 0$ in $\mathscr{S}$ and $p \in \mathscr{P}$ such that $s = u - v$, $u(p) < \epsilon$, and $v(p') < \epsilon$.

4. Theorem. Let Δ be a convex, full and unital set of states on Π. If $\mathscr{P} \subseteq \Pi$ is Δ fundamental then the following are equivalent:

(1) For each $s \in \mathscr{S}$, $\|s\|_Z = \|s\|$.

(2) The $w(\mathscr{S}^*, \mathscr{S})$-closed convex extension of $\{p - p' : p \in \mathscr{P}\} = \{\phi \in \mathscr{S}^* : -e \leqslant \phi \leqslant e\}$; alternatively expressed, $\{p - p' : p \in \mathscr{P}\}_\circ{}^\circ = [-e, e]$.

(3) $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated.

Proof. (1) iff (2): we have $\|s\|_Z = \|s\|$ for all $s \in \mathscr{S}$ iff their respective dual unit balls in $\mathscr{S}^*$ are the same. These are, respectively, $\{p - p' : p \in \mathscr{P}\}_\circ{}^\circ$ and $[-e, e]$.

(1) implies (3): let $s \in \mathscr{S}$; to demonstrate (3) we may assume without loss that neither $s \geqslant 0$ nor $-s \geqslant 0$ and $\|s\| = 1$. (If $s \geqslant 0$, $s = s - 0$ and for $p = 0$, $s(p) = 0(p') = 0$.) Let $\epsilon > 0$ be given. Then there exist $p \in \mathscr{P}$, scalars $\alpha, \beta > 0$, and ω, ν in Δ such that $s = \alpha\omega - \beta\nu$, $1 + \epsilon/2 > \alpha + \beta \geqslant 1 = \|s\|$, and $s(p - p') + \epsilon/2 > 1 = \|s\|_Z$. Thus, $s(p - p') + \epsilon = s(2p - e) + \epsilon > \alpha + \beta \geqslant 1$. This implies $\alpha\omega(p) - \beta\nu(p) + \epsilon/2 > \alpha$; hence, $\epsilon/2 > \alpha\omega(p') + \beta\nu(p)$. Let $u = \alpha\omega$ and $v = \beta\nu$. We have $s = u - v$, $u(p') < \epsilon$ and $v(p) < \epsilon$ and (3) is demonstrated.

(3) implies (1): assume neither $s \geqslant 0$ nor $-s \geqslant 0$ and $\epsilon > 0$ is given. As already observed $\|s\|_Z \leqslant \|s\|$ for all s in $\mathscr{S}$. If (3) holds then there exist scalars $\alpha, \beta > 0$, ω, ν in Δ and $p \in \mathscr{P}$ such that $s = \alpha\omega - \beta\nu$, $\alpha\omega(p') < \epsilon$ and $\beta\nu(p) < \epsilon$. Then, $s(p - p') = \alpha\omega(2p - e) - \beta\nu(e - 2p') = 2[\alpha\omega(p) + \beta\nu(p')] - (\alpha + \beta)$. Since $\omega(p') < \epsilon/\alpha$, $\omega(p) = 1 - \omega(p') > 1 - \epsilon/\alpha$; similarly, $\nu(p') > 1 - \epsilon/\beta$. Thus $\|s\|_Z \geqslant s(p - p') > 2[\alpha(1 - \epsilon/\alpha) + \beta(1 - \epsilon/\beta)] - (\alpha + \beta) = \alpha + \beta - 4\epsilon$. Hence, $\|s\|_Z \geqslant \|s\|$. ∎

Let E be a locally convex space with dual E^*. If M is a subset of E^* (not necessarily a subspace), we denote the weakest linear topology on E making each element of M continuous by $w(E, M)$. We note that $w(E, M) = w(E, \text{span}(M)) \subseteq w(E, E^*)$. The topology $w(E, M)$ is Hausdorff if M separates the points of E.

5. Theorem. Let Π be a σ-complete lattice, Δ a convex, full, and unital set of states, and let $\mathscr{P}$ be a Δ-fundamental subset of Π. If Δ is $w(\mathscr{S}, \mathscr{P})$-sequentially compact, then the following are equivalent:

(1) $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated.

(2) $\mathscr{S}$ is Π-Jordan generated.

Proof. (2) implies (1) is clear. To prove the converse let $s \in \mathcal{S}$ and suppose, for simplicity, that $\|s\| = 1$. For each positive integer k there exist scalars $\alpha_k, \beta_k \geqslant 0$, and $\omega_k, \nu_k \in \Delta$ such that $s = \alpha_k \omega_k - \beta_k \nu_k$ and $1 + 1/k \geqslant \alpha_k + \beta_k \geqslant 1 = \|s\|$. Since Δ is $w(\mathcal{S}, \mathcal{P})$-sequentially compact there exist $\omega, \nu \in \Delta$ such that (selecting subsequences if necessary) $\omega_k \to \omega$ and $\nu_k \to \nu$. Similarly, $\alpha_k \to \alpha$ and $\beta_k \to \beta$ in $[0, 1]$ and $1 = \alpha + \beta$. Thus, $s = \alpha\omega - \beta\nu$. We may assume α and $\beta \neq 0$. Since we are assuming (1) there exist $(p_k) \subset \mathcal{P}$ such that (A) $1 \geqslant \alpha\omega(2p_k - e) - \beta\nu(2p_k - e) > 1 - 1/k$. Hence, $1 \geqslant \alpha[2\omega(p_k) - 1] + \beta[1 - 2\nu(p_k)] > 1 - 1/k$ and therefore, either $1 - 2\nu(p_k) > 1 - 1/k$ or $2\omega(p_k) - 1 > 1 - 1/k$. Without loss assume the former; this implies $\nu(p_k) < 1/2k$. If $\bigwedge_k p_k = p \in \Pi$ then $\nu(p) = 0$, since $\nu \in \Delta$ and Π is a σ-complete orthomodular lattice.

From (A) follows: $1 \geqslant \alpha[2\omega(p) - 1] + \beta \geqslant 1$. Hence, $2\omega(p) - 1 = 1$ and $\omega(p') = 0$. ∎

We now observe that ω and ν are mutually singular and we have obtained a close analog to the standard Hahn–Jordan decomposition for signed measures.

6. Corollary. Let Δ be a convex, full, and unital set of states, $\mathcal{P}$ be Δ-fundamental in Π, and let Δ be $w(\mathcal{S}, \mathcal{P})$-sequentially compact; then $(\mathcal{S}, \|\cdot\|)$ is a Banach space.

Proof. Let (s_n) be a norm-Cauchy sequence in $\mathcal{S}$. We may assume without loss that $\|s_n\| \leqslant 1$ for all n. Following the proof of the previous theorem we have $s_n = \alpha_n \omega_n - \beta_n \nu_n$, $\alpha_n, \beta_n \geqslant 0$, $\alpha_n + \beta_n \leqslant 1$, and $\omega_n, \nu_n \in \Delta$. Again, selecting subsequences if necessary, let $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\omega_n \to \omega$, and $\nu_n \to \nu$. Then $s_n \to s = \alpha\omega - \beta\nu$ in $w(\mathcal{S}, \mathcal{P})$. Since $\mathcal{P}$ is total and (s_n) is norm-Cauchy, we have norm convergence as in Proposition 2. ∎

A few remarks concerning the paper of Kronfli (1970) are in order. The "natural metric" of Kronfli $d(\omega, \nu) = \sup\{\omega(p) - \nu(p) : p \in \Pi\}$ is clearly $\frac{1}{2}\|\omega - \nu\|_Z$ on Ω. The "total variation" norm of $\mathcal{S}$ as defined there agrees with the base norm when Kronfli's positive and negative parts are in Ω. Since these parts may not be orthogonally additive, it would be interesting to know what hypothesis must be added to this discussion in order that they be additive. If this were so we would have a nearly complete generalization of the Hahn–Jordan theorem of measure theory.

We now take up a final refinement of these concepts. This result is a reformulation of a result of Dixmier (1948, p. 1069) and Edwards (1964, p. 405) to generalized quantum logics.

7. Theorem. Let Δ be a convex, full, and unital set of states and let $\mathcal{P}$ be a Δ-fundamental set of propositions in Π. If $\mathcal{S}$ has the base-normed topology and $\mathcal{S}_*$ is the norm-closure of span $(\mathcal{P})$ in $\mathcal{S}^*$, then $\mathcal{S}_*$ is an order-unit Banach subspace of the order-unit

Banach space $\mathscr{S}^*$. If $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated then $\mathscr{S}$ is order-isometric to a $w(\mathscr{S}_*^*, \mathscr{S}_*)$-dense subspace of the base-normed dual space $\mathscr{S}_*^*$. In particular, $\mathscr{S}$ is order-isometric with $\mathscr{S}_*^*$ iff Δ is $w(\mathscr{S}, \mathscr{P})$-compact.

Proof. $\mathscr{S}^*$ is an order-unit Banach space. Clearly, $\mathscr{S}_*$ is also an order-unit Banach subspace of $\mathscr{S}^*$. The ordered Banach space $\mathscr{S}_*^*$ is a base-normed dual space of $\mathscr{S}_*$; we denote its base by $\tilde{\Delta}$. If we consider the dual pair $(\mathscr{S}_*, \mathscr{S})$ then $\mathscr{S}_*$ separates the points of $\mathscr{S}$. To see this, suppose $\phi(s) = 0$ for some s in $\mathscr{S}$ and all ϕ in $\mathscr{S}_*$. Then $s(p) = 0$ for all $p \in \mathscr{P}$ and from the hypothesis that $\mathscr{S}$ if ϵ–$\mathscr{P}$-Jordan generated it follows that s is the zero functional on $\mathscr{S}^*$. Hence, $\mathscr{S}$ is isomorphic to a $w(\mathscr{S}_*^*, \mathscr{S}_*)$-dense subspace of $\mathscr{S}_*^*$. Henceforth we will simply consider $\mathscr{S} \subseteq \mathscr{S}_*^*$. It is clear that the order of $\mathscr{S}$ is the same as that induced from $\mathscr{S}_*^*$. The unit ball U of $\mathscr{S}_*^*$ is the polar of $\{p - p' : p \in \mathscr{P}\}$. Consequently, the unit ball of $\mathscr{S}$ is $w(\mathscr{S}_*^*, \mathscr{S}_*)$-dense in U with Δ being $w(\mathscr{S}_*^*, \mathscr{S}_*)$-dense in $\tilde{\Delta}$.

If $\mathscr{S} = \mathscr{S}_*^*$, by the Banach–Alaoglu theorem, Δ is $w(\mathscr{S}, \mathscr{S}_*)$-compact. Hence Δ is also $w(\mathscr{S}, \mathscr{P})$-compact, since $\mathscr{P} \subset \mathscr{S}_*$. Conversely, if Δ is $w(\mathscr{S}, \mathscr{P})$-compact then Δ is $w(\mathscr{S}, \mathscr{S}_*)$-compact, since the $w(\mathscr{S}, \mathscr{S}_*)$-topology agrees on Δ with the $w(\mathscr{S}, \mathscr{P})$-topology, see Robertson and Robertson (1966, Cor. 3, p. 104). Consequently, Δ is $w(\mathscr{S}_*^*, \mathscr{S}_*)$-compact and $\Delta = \tilde{\Delta}$; thus $\mathscr{S} = \mathscr{S}_*^*$. ∎

We would now like to demonstrate that each proposition in Π is an extremal point for $[0, e]$. Since $[-e, e]$ is $w(\mathscr{S}^*, \mathscr{S})$-compact and convex and affinely homeomorphic to $[0, e]$, this is equivalent to showing each $p - p'$ is extremal in $[-e, e]$ for each $p \in \Pi$. If $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated, then, using Milman's theorem, we know that the extreme points of $[0, e]$ are in the w^*-closure of $\mathscr{P}$. Recall Milman's theorem: Suppose M is a set whose closed convex cover N is compact. The extreme points of N lie in the closure of M (Köthe, 1969, p. 332). In order to demonstrate that each proposition is extremal, we need to add some hypotheses to Δ.

If $\Delta \subset \Omega$, we say Δ is a *strong* set of states provided for each pair, p, q in Π: $\{\omega \in \Delta : \omega(p) = 1\} \subseteq \{\omega \in \Delta : \omega(q) = 1\}$ implies $p \leqslant q$ in Π. We observe that if Δ is strong then Δ is separating, full, and unital.

8. Theorem. If Δ is strong over Π, then each $p \in \Pi$ is extreme in the convex hull of Π (denoted $\langle \Pi \rangle$) in $\mathscr{S}^*$.

Proof. Let $p_0 \in \Pi$ and suppose p_0 belongs to the interior of a line segment joining two elements of $\langle \Pi \rangle$; i.e.,

$$p_0 = \sum_{i=1}^{n} \alpha_i p_i, \qquad p_i \in \Pi, \qquad \alpha_i > 0$$

and

$$\sum_{i=1}^{n} \alpha_i = 1$$

If $\omega \in \Delta$ and $\omega(p_0) = 1$ then $\omega(p_i) = 1$, $i = 1, 2, \ldots, n$. Since Δ is strong, $p_0 \leqslant p_i$ in $\mathscr{S}^*$, $i = 1, 2, \ldots, n$. Suppose for some i, say $i = 1$, there exists some $\omega_0 \in \Delta$ such that $\omega_0(p_0) < \omega_0(p_1)$. Then,

$$\omega_0(p_0) = \sum_{i=1}^{n} \alpha_i \omega_0(p_i) > \alpha_1 \omega_0(p_0) + \sum_{i=2}^{n} \alpha_i \omega_0(p_i)$$

or

$$\omega_0(p_0) > \sum_{i=2}^{n} \alpha_i/(1 - \alpha_1)\omega_0(p_i)$$

Since $p_0 \leqslant p_i$, $\alpha_i/(1 - \alpha_1) > 0$, $i = 2, \ldots, n$, and

$$\sum_{i=2}^{n} \alpha_i/(1 - \alpha_1) = 1$$

it must also be true that

$$\omega_0(p_0) \leqslant \sum_{i=2}^{n} \alpha_i/(1 - \alpha_1)\omega_0(p_i)$$

This gives a contradiction, so $p_0 = p_1 = \cdots = p_n$. ∎

9. Proposition. Let Δ be strong over Π, $p, q \in \Pi$ and $\alpha > 0$ in $\mathbb{R}$. If $\alpha p \leqslant q$ in $\mathscr{S}^*$ then $p \leqslant q$ in $\mathscr{S}^*$ and, of course, in Π.

Proof. Let $\omega \in \Delta$ such that $\omega(q') = 1$. This implies $\omega(q) = 0$ and thus $\omega(p) = 0$. Whence, $\omega(p') = 1$. Since Δ is strong over Π, $q' \leqslant p'$ and therefore $p \leqslant q$. ∎

If $\mathscr{P}$ is a Δ-fundamental subset of Π and $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated, then for each $\phi \in [0, e]$ there is a net (ϕ_β) in the convex hull of $\mathscr{P}$ (denoted $\langle\mathscr{P}\rangle$) such that $\phi = \text{weak*} - \lim_\beta (\phi_\beta)$. If, in addition, this net can be chosen so that $\phi_\beta \leqslant \phi$ for all β, then we will say that $\mathscr{S}$ is *monotonically* ϵ–$\mathscr{P}$-Jordan generated. This is a far weaker assumption than having each element of $\mathscr{S}^*$ represented as a spectral integral.

10. Theorem. Suppose Δ is strong over Π, $\mathscr{P}$ is a Δ-fundamental subset of Π, and $\mathscr{S}$ is monotonically ϵ–$\mathscr{P}$-Jordan generated. Then each $p \in \Pi$ is an extreme point of $[0, e]$ and each extreme point of $[0, e]$ is in the $w(\mathscr{S}^*, \mathscr{S})$ closure of $\mathscr{P}$.

Proof. Let $p \in \Pi$, $\phi_1, \phi_2 \in [0, e]$, and let $p = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$. We show that $p = \phi_1 = \phi_2$ by showing that $\omega(p) = \omega(\phi_1) = \omega(\phi_2)$ for each $\omega \in \Delta$.

Let $\omega \in \Delta$ and $\epsilon > 0$ be given. Then there exist $\psi_1, \psi_2 \in \langle \mathscr{P} \rangle$ such that $\psi_1 \leqslant \phi_1$, $\psi_2 \leqslant \phi_2$, $0 \leqslant \omega(\phi_1 - \psi_1) < \epsilon/2$, and $0 \leqslant \omega(\phi_2 - \psi_2) < \epsilon/2$. Let

$$\psi_i = \sum_{k=1}^{n_i} \beta_{k,i} p_{k,i}$$

for $i = 1, 2$ and these are understood to be nontrivial convex combinations. For $i = 1$, $\frac{1}{2}\psi_1 \leqslant p$ implies $\frac{1}{2}\beta_{k,1} p_{k,1} \leqslant p$, $1 \leqslant k \leqslant n_1$. By Proposition 9 each $p_{k,1} \leqslant p$. A similar computation for $i = 2$ can be made and therefore we have $\psi_1, \psi_2 \leqslant p$. Then, $0 \leqslant \omega[p - (\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2)] \leqslant \frac{1}{2}\omega(\phi_1 - \psi_1) + \frac{1}{2}\omega(\phi_2 - \psi_2) < \epsilon/2$. Therefore, $0 \leqslant \omega[p - (\frac{1}{2}\psi_1 + \frac{1}{2}\psi_2)] = \frac{1}{2}\omega(p - \psi_1) + \frac{1}{2}\omega(p - \psi_2) < \epsilon/2$. Thus, $\omega(p - \psi_i) < \epsilon$, $i = 1, 2$; it follows: $|\omega(p - \phi_i)| \leqslant |\omega(p - \psi_i)| + |\omega(\psi_i - \phi_i)| < 2\epsilon$. Consequently, $\omega(p) = \omega(\phi_1) = \omega(\phi_2)$. ∎

11. Corollary. When, in addition, Δ is $w(\mathscr{S}, \mathscr{P})$-compact in the theorem above, each element of $\mathscr{P}$ is an extreme point for the unit interval $[0, e]$ in $\mathscr{S}_*$ and the unit ball of $\mathscr{S}_*$ is the $w(\mathscr{S}_*, \mathscr{S})$-closed (also norm-closed) convex hull of $\{p - p' : p \in \mathscr{P}\}$.

Note that $[0, e]$ in $\mathscr{S}_*$ may not be compact in any locally convex Hausdorff topology. (Consider the Hilbert space example in the next section.)

4. EXAMPLES

The first example we develop is the standard model of nonrelativistic quantum mechanics. Let H be a separable, infinite-dimensional, complex Hilbert space. The logic Π is taken to be the set of all orthogonal projections on H. The identity operator $\mathscr{I}$ is identified with e in Π and the partial order of Π is the following: $P \leqslant Q$ in Π if $R(P)$ (the range of P) $\subseteq R(Q)$. Orthogonality is defined by $P \perp Q$ if $R(P) \perp R(Q)$ in H. It is well known that $(\Pi, \leqslant, \perp)$ is a σ-complete orthomodular lattice, see for example, Jauch (1968, Chapter 5). Using Gleason's theorem (1957) we may identify all the countably additive states with the positive self-adjoint trace class operators on H of unit trace. Again we denote this set with Δ. Then span $(\Delta) = \mathscr{S}$ is the real partially ordered base-normed space of self-adjoint trace class operators. Recall that the ordering is given by $A \in \mathscr{S}$, $A \geqslant 0$ if $(Ax, x) \geqslant 0$ for all $x \in H$ where $(\cdot, \cdot)$ denotes the inner product in H.

The order unit dual $\mathscr{S}^*$ of $\mathscr{S}$ is order-isometrically identified with the Banach space $B(H)$ of all bounded self-adjoint operators on H. If $A \in B(H)$ and $C \in \mathscr{S}$ then $\text{Tr}\,(C \circ A) = \text{Tr}\,(A \circ C)$ gives the bilinear pairing of $\mathscr{S}$ and $B(H)$ where $\text{Tr}\,(\circ)$ denotes the trace of an element of $\mathscr{S}$. For the details of this construction see Schatten (1960, p. 46).

Let $x, y \in H$. Define the operator $x \otimes \bar{y}$ on H as follows: for $z \in H$, $x \otimes \bar{y}(z) = (z, y)x$. Let P be a projection on H with range $R(P)$. If (e_n) is any orthonormal basis for $R(P)$ then $P = \sum_n e_n \otimes \bar{e}_n$ where this series converges monotonically in the $w(\mathscr{S}^*, \mathscr{S})$-topology, see Schatten (1960, p. 10). In particular, for each $x \in H$, $P(x) = \sum_{n=1}^{\infty} (x, e_n)e_n$. Finally, note that for each $x \in H$ with $\|x\| = 1$, $x \otimes \bar{x} \in \Delta$, since $\mathrm{Tr}\,(x \otimes \bar{x}) = (x, x) = 1$.

We now observe Δ is strong over Π: if $P, Q \in \Pi$, $\omega \in \Delta$ and $\omega(P) = 1$ implies $\omega(Q) = 1$ then we show $R(P) \subseteq R(Q)$ and, thus, $P \leqslant Q$. Let $e_0 \in R(P)$ and $P = \sum_n e_n \otimes \bar{e}_n$; then $\mathrm{Tr}\,(e_0 \otimes \bar{e}_0 \circ P) = \mathrm{Tr}\,(e_0 \otimes \bar{e}_0) = 1$. Hence $\mathrm{Tr}\,(e_0 \otimes \bar{e}_0 \circ Q) = 1$. Consequently, $e_0 \in R(Q)$.

We next show that $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated when $\mathscr{P}$ is chosen as the fundamental set of all projections with the property $p \in \mathscr{P}$ if $R(P)$ is finite dimensional or $p = \mathscr{I}$. We loosely refer to these as the finite projections of H. It is easy to see that $\mathscr{P}$ is indeed Δ-fundamental. Recall that if $\omega \in \Delta$ then $\omega = \sum_n \lambda_n e_n \otimes \bar{e}_n$ where (e_n) is an orthonormal sequence, $\lambda_n > 0$, and $\sum_n \lambda_n = 1$, see Schatten (1960, p. 41). The cone of $\mathscr{S}$ is generating and the representation of ω is norm-convergent to ω. If $x \in \mathscr{S}$, using the spectral theorem, x may be written as $x = \alpha\omega_1 - \beta\omega_2$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = \|x\|$, and $\omega_i = \sum_n \lambda_n^{(i)} e_n^{(i)} \otimes \bar{e}_n^{(i)} \in \Delta$ for $i = 1, 2$ and with $e_n^{(1)} \perp e_k^{(2)}$ for all n, k. For each $\epsilon > 0$ there is an index N such that $P = \sum_{n=1}^{N} e_n^{(1)} \otimes \bar{e}_n^{(1)} \in \mathscr{P} \subseteq \Pi$ and $\mathrm{Tr}\,(\omega_1 \circ P) \geqslant 1 - \epsilon$. Hence, $\mathrm{Tr}\,((\mathscr{I} - P) \circ \omega_1) < \epsilon$ and $\mathrm{Tr}\,(\omega_2 \circ P) = 0$.

If we now consider the norm closed linear span of $\mathscr{P}$ in $B(H)$ and call this space $\mathscr{S}_*$, then $\mathscr{S}_*$ is the order unit space of self-adjoint compact operators with the identity operator adjoined. It is well known that $(\mathscr{S}_*)^*$ is order-isometric to $\mathscr{S}$, see Schatten (1960, p. 46). Consequently, Δ is $w(\mathscr{S}, \mathscr{P})$-compact and, of course, $\mathscr{S}$ is a Banach space.

Lastly, to show that each $P \in \Pi$ is extremal in $[0, \mathscr{I}]$, we show for each $A \in [0, \mathscr{I}]$, there exists a net $(\phi_\alpha) \subset \langle \mathscr{P} \rangle$ such that $w^* - \lim \phi_\alpha = A$ and each $\phi_\alpha \leqslant A$. We have already observed that if $P \in \Pi$, $P = \sum_n e_n \otimes \bar{e}_n$ and this series converges monotonically from below in the w^* topology. Thus, for each $\epsilon > 0$ and $\omega \in \Delta$ there exists a finite projection $P_1 \in \mathscr{P}$ with $P_1 \leqslant P$ and $\omega(P - P_1) < \epsilon$.

Let Λ be the spectrum of A. Then Λ is a compact subset of $[0, 1]$ in $\mathbb{R}$. Without loss we may assume $\|A\| = 1$, so that $1 \in \Lambda$. From the Gelfand representation theorem, the closure of the real polynomial algebra generated by A in $B(H)$ is lattice isometric to the Banach lattice $C(\Lambda, \mathbb{R})$. Under this mapping $p(\lambda) = \lambda$ corresponds to A and $1 = \chi_\Lambda$ corresponds to $\mathscr{I}$ in $B(H)$. This mapping can be extended so that the characteristic function of each Borel set in Λ corresponds to a projection in $B(H)$. Let $\Gamma_k = [k/n, 1] \cap \Lambda$ for $k = 1, 2, \ldots, n$ and let $\Gamma_k \rightsquigarrow P_k \in \Pi \subset B(H)$. Then,

$$\frac{1}{n} \geqslant \left\| \lambda - \sum_{k=1}^{n} \frac{1}{n} \chi_{\Gamma_k}(\lambda) \right\| \geqslant \left\| A - \sum_{k=1}^{n} \frac{1}{n} P_k \right\|,$$

see Reed and Simon (1972, p. 225). Since A is norm approximated from below in the interval $[0, \mathscr{I}]$ by $\sum_{k=1}^{n} (1/n)P_k$, the desired net, which is to be w^* convergent to A, can be constructed. Thus, using Theorem 10, we have shown that each projection on H is extremal for $[0, \mathscr{I}]$ in $B(H)$ (Kadison, 1951, p. 328). In this situation one can actually show that all extreme points of $[0, \mathscr{I}]$ are projections, but this is not directly deducible from Theorem 10.

Let us now consider the second example. Let Y be a compact Hausdorff space. Let Π be the family of all Baire sets of Y. By the Baire sets we mean the smallest σ algebra of subsets of Y that contains the compact-G_δ sets. We may define orthogonality in Π by disjointness in Y; i.e., $B_1, B_2 \in \Pi$ and $B_1 \perp B_2$ means $B_1 \cap B_2 = \phi$ in Y. The order $\leqslant$ on Π is set inclusion and $(\Pi, \leqslant, \perp)$ is a σ-complete Boolean lattice. Let Δ be the set of all Baire-probability measures on Π. The space $\mathscr{S}$ is the Banach space of all signed Baire measures with the total variation norm. This is the same as the base-norm topology on $\mathscr{S}$. Note that $C(Y)^* = \mathscr{S}$.

Each Baire set $B \subset Y$ defines a linear functional p_B on $\mathscr{S}$ by the formula $p_B(s) = \int \chi_B \, ds$ for each $s \in \mathscr{S}$. Hence, it is easy to see that Π is order injected into $\mathscr{S}^*$. The state space Δ is strong over Π: Suppose $p_A, p_B \in \Pi$ and for each $\mu \in \Delta$, $\mu(p_A) = 1$ implies $\mu(p_B) = 1$. If $a \in A$ and $\delta_a \in \Delta$ is the unit-point measure at a, then $\int \chi_A \, d\delta_a = 1$ and this implies $\int \chi_B \, d\delta_a = 1$. Hence $a \in B$ and $p_A \leqslant p_B$.

$\mathscr{S}$ is Π-Jordan generated. This is simply the classical Hahn–Jordan decomposition theorem for signed measures.

Several choices for $\mathscr{P} \subseteq \Pi$ now exist. These choices will depend upon the topological structure of Y. Since we are assuming, at this point, only that Y is compact, let us choose $\mathscr{P} = \{p_D: D \text{ a closed Baire set}\}$. Then $\mathscr{P}$ is a fundamental family: Clearly, $0, e \in \mathscr{P}$ and $\mathscr{P}$ is closed under finite joins. Also, $\mathscr{P}$ is total over Δ: Suppose $s \in \mathscr{S}$, $\|s\| = 1$ and $s(p) = 0$ for all $p \in \mathscr{P}$. Then $s = \alpha\mu_1 - \beta\mu_2$, $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$ and $\mu_1, \mu_2 \in \Delta$. If $\alpha \neq 0$, $\mu_1 = (\beta/\alpha)\mu_2$ on all compact-G_δ sets in Y. Since μ_1 and μ_2 are Baire measures, we have $s = 0$ in $\mathscr{S}$. Since each Baire measure is regular, we have that $\mathscr{P}$ is fundamental.

Next, $\mathscr{S}$ is ϵ–$\mathscr{P}$-Jordan generated. This follows from the regularity of each Baire measure. If $\mu \in \mathscr{S}$ and $\|\mu\| = 1$, then there exist two mutually singular Baire measures μ_1 and μ_2 ($\|\mu_i\| \leqslant 1, i = 1, 2$) such that $\mu = \mu_1 - \mu_2$. Let T_1 and T_2 be the closed Baire sets which support μ_1 and μ_2, respectively. For $\epsilon > 0$ there exist compact-G_δ sets $V_1 \subseteq T_1$ and $V_2 \subseteq T_2$ such that $V_1 \cap V_2 = \phi$ and $\mu_2(V_1) = \mu_1(V_2) = 0 < \epsilon$.

Denote the norm-closed linear span of $\mathscr{P}$ in $\mathscr{S}^*$ by $\mathscr{S}_*$. The base Δ is $w(\mathscr{S}, C(Y))$-compact. Let D be a Baire set and $p_D \in \mathscr{S}^*$. Then $\chi_D \in C(Y)$ iff D is a clopen set in Y. Thus for a connected compact space Y, Δ is not $w(\mathscr{S}, \mathscr{S}_*)$-compact but is $w(\mathscr{S}, \mathscr{S}_*)$-closed. For general results like this see Edwards (1964, p. 405).

To see that each $p_D \in \Pi$ corresponds to an extreme point in $[0, e]$ in $C(Y)^{**}$, we observe that $C(Y)^{**}$ is lattice-isometric to a space $C(Z)$ where Z is a hyperstonian-compact space and p_D is associated to a characteristic function of a clopen set of Z. For details of this one can see, for example, Kelley and Namioka (1963, §24), or Bade (1971, Section 8). Using the structure of $C(Y)^{**}$ one can also obtain this result from Theorem 10 of this paper.

Finally, if $\mathscr{B}$ is a Boolean algebra, then, via Stone's representation theorem, $\mathscr{B}$ may be identified with the clopen sets of a totally disconnected compact Hausdorff space Y. Each clopen set is a Baire set and in this case we may take $\mathscr{P} = \{p_D: D \text{ clopen in } Y\}$. Since the clopen sets of Y form a basis for Y, we may now apply the second example directly to this case. We observe that $C(Y) = \mathscr{S}_*$, Δ is $w(\mathscr{S}, \mathscr{P})$-compact and the unit ball of $C(Y)$ is not compact (in general) but is the norm-closed convex hull of its extreme points.

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